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ENGINEERING RELATIONSHIPS FOR TURBULENT FORCED-CONVECTION HEAT TRANSFER IN DUCTS WITH FLUX TRANSIENTS

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NOMENCLATURE

a	Width of Annular Space
A	Heat-transfer area per unit axial length of duct
c	Specific heat of fluid
D	Hydraulic equivalent diameter
E_n	Eigenfunction
F_i	Dimensionless heat flux, $q_i a/k\Delta t_r$
F_i^*	Dimensionless heat flux corresponding to ξ^*
g	Dimensionless fluid velocity, $u/\bar{u}$
h	Heat-transfer coefficient
k	Thermal conductivity
ℓ	Axial distance from duct inlet
L	Differential operator defined by Eq. 6
N_n	Normalization factor
Nu	Nusselt number, Dh/k
Pe	Peclét number, $D\bar{u}/\alpha$
Pr	Prandtl number, ν/α
q	Wall heat-flux density
r_1	Inner radius of annular space
r_2	Outer radius of annular space
R	Annulus radius ratio, r_1/r_2
Re	Reynolds number, $D\bar{u}/\nu$
$R_{i,k}$	Coefficient defined by Eqs. 33a and 33c
S	Cross-sectional area of duct
$S_{i,k}$	Coefficient defined by Eqs. 33b and 33d
t	Fluid temperature
t_0	Fluid temperature at duct inlet
t_s	Duct wall temperature
$t_{s,i}$	Surface temperature of inner or outer wall of annular space
$\bar{t}$	Fluid bulk temperature
u	Fluid velocity
$\bar{u}$	Average fluid velocity
x	Dimensionless distance from inner wall of annular space, y/a
y	Distance from inner wall of annular space
z	Dimensionless axial distance $(4/\mathrm{Pe})(\ell/D)$
Z	Dimensionless axial distance, $\beta(\ell/D)$
α	Fluid thermal diffusivity
β	Dimensionless coefficient, $4\lambda_1/\mathrm{Pe}$
Δt_r	Arbitrary reference temperature difference
ϵ	Eddy diffusivity for heat transfer
$\zeta_{i,k}$	Function defined by Eq. 16
$\zeta'_{i,k}$	Function defined by Eq. 16 with $g(x) = 1$
θ	Dimensionless time $(4/\mathrm{Pe})(\bar{u}\tau/d)$
Θ	Dimensionless time, $\beta(\bar{u}\tau/D)$
λ_n	Eigenvalue
λ_1	Least nonzero eigenvalue
ν	Kinematic viscosity
ξ	Dimensionless fluid temperature $(t - t_0)/\Delta t_r$
ξ^*	Dimensionless fluid initial temperature
ρ	Fluid density
τ	Time
ψ	Dimensionless fluid temperature for uniform initial conditions, $\xi - \xi^*$

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ABSTRACT

Engineering relationships to account for turbulent forced-convection heat transfer in ducts during heat-flux transients and steady flow are derived. These relationships are suggested as more accurate replacements for the simple heat-transfer-coefficient expression $t_s - \bar{t} = q/h$ currently used in most engineering computations involving transients. The new relationships have the form

$$t_s - \bar{t} = \frac{q}{h} + \frac{D^2 R_1}{2k\beta}\left(\frac{\partial q}{\partial \ell} + \frac{1}{\bar{u}}\frac{\partial q}{\partial \tau}\right) + \dots,$$

where R_1 and β are dimensionless coefficients depending on Reynolds and Prandtl numbers and on duct shape. Tabulated values of these coefficients for the circular tube, parallel-plane duct, and pin bundle are given for wide ranges of Reynolds and Prandtl numbers, with emphasis on liquid metals.

INTRODUCTION

The necessity to account for turbulent forced-convection heat transfer in ducts during wall heat-flux transients occurs frequently in engineering analysis. One of the oldest problems of this kind occurs in the design of periodic-flow heat-exchangers (regenerators); one of the more recent occurs in safety studies of liquid-metal fast-breeder nuclear reactors, where it is required to consider incidents involving large and extremely rapid heat-flux changes. Although the main motivation for the work reported here is nuclear-reactor applications, with emphasis on liquid metals, the results have more general applicability.

The traditional engineering method used to account for forced-convection heat transfer during transients assumes that the simple heat-transfer-coefficient relation applies, i.e., that the local heat flux at the duct wall is directly proportional to the temperature difference between the wall and the "bulk" of the fluid. Thus, the familiar relationship

$$t_s - \bar{t} = q/h \tag{1}$$

is used in most, if not all, practical engineering analyses involving transients. In general, q, t_s, and $\bar{t}$ are unknown functions of time and duct axial distance; and h is taken equal to its fully developed steady-state value, and is independent of time when the flow is steady.

Equation 1 serves as a coupling or boundary condition between an appropriately simplified version of the energy equation applied to the fluid, and heat conduction in the duct walls or its equivalent. For example, for a duct of sufficiently simple cross section, and with no internal heat sources within the fluid, the relationship

$$c\rho S\left(\bar{u}\,\frac{\partial \bar{t}}{\partial \ell} + \frac{\partial \bar{t}}{\partial \tau}\right) = Aq \tag{2}$$

would be used for the fluid. In fact, the use of Eq. 1 implies the use of Eq. 2, since together they represent a one (space)-dimensional approximation to the exact energy equation for the fluid.

For regenerators, ducts with specified time-dependent heat generation within the walls, and other similar devices, the heat-conduction equation is also required to account for transient heat transfer within the duct walls, thereby relating t_s and q. This additional requirement results in a complicated mathematical problem, which, except for certain special limiting cases, can be solved only by numerical methods and machine computation. If the simplifications afforded by Eqs. 1 and 2 were eliminated by direct use of a more exact representation of the energy equation for the fluid, the necessary machine computations would become cumbersome and time-consuming.

For nuclear reactors, internal heat generation in the equivalent of the duct walls requires, in addition to the transient heat-conduction equation, introduction of appropriate neutron-kinetic expressions, which, in turn, are influenced by important interacting temperature effects. Elimination of the approximations implicit in the use of Eqs. 1 and 2 by direct use of a more exact version of the energy equation for the fluid is probably impractical today.

Thus, for analyses of complex systems such as nuclear reactors, and probably also for simpler devices such as regenerators, the use of Eq. 1 to account for turbulent forced convection during transients offers a simplicity which is necessary in the interests of computational practicability. The potential for inaccuracies resulting from the use of Eq. 1 remains, of course; concern for these inaccuracies has existed in the nuclear-reactor field for some time.[1-4]

Various analytical investigations related to more exact predictions of convection heat transfer with wall-heat-flux transients in ducts have been reported in the literature. All the investigations concentrate on specific cases; most consider idealizations in which the wall heat flux or surface temperatures are specified functions of time;[5-9] and a few consider cases in which heat generation within a duct wall is specified.[10-11] All these publications emphasize mathematical techniques and approximations for solutions of the specific cases considered; and comparisons of results give some information related to the accuracy of the simpler calculations based on the use of Eqs. 1 and 2. However, the information is specific to the cases considered, and definitive generalizations are not possible.

Perhaps the most recent and most detailed of these analytical investigations was performed by Gopalakrishnan,[11] who treated the case of turbulent flow through a parallel-plate duct with transient internal heat generation within the duct walls. He developed a general computer program based on finite-difference representations of the coupled two (space)-dimensional energy equations for the fluid and the duct walls, and then applied the program to various specific cases involving exponentially increasing or decreasing wall heat generation. He found that for sufficiently large time and distance from the duct inlet, h in Eq. 1 became constant, i.e., attained an asymptotic value. He also found, however, that for sufficiently rapid heat-generation transients, the asymptotic value of h depends on the exponential period of the transient, and that this dependence occurred in ranges of interest to nuclear-reactor applications. Thus, for such cases, even when asymptotic values are obtained, the usual steady-state heat-transfer coefficient is not correct for use in Eq. 1. Instead, an appropriate value depends on the form of the transient, which, of course, is usually not known in advance.

In general, all the referenced investigations can be interpreted as showing that errors resulting from the use of Eq. 1 can be quite large in many cases of engineering interest. None of the investigations has found potentially more accurate alternates to Eq. 1, other than the equivalent of using a more exact version of the fluid-energy equation to replace both Eqs. 1 and 2.

Investigations emphasizing attempts to correlate transient experimental data are being performed by researchers in the Soviet Union.[12] These investigations are not focused on mathematical solutions of specific cases; rather they are based on analytical and physical reasoning which suggests that h in Eq. 1 can be correlated empirically with experimental data, provided additional dimensionless parameters are included with those usually associated with nontransient convection heat transfer. For heat-flux transients, the pertinent dimensionless parameter suggested is proportional to the time rate of change of duct surface temperature.

For example, the recent publication of Koshkin et al.[13] gives empirical correlations of turbulent gas-flow transient heat-transfer coefficients from experiments with step changes of electrical heat generation in the walls of a tube. The generality of these types of correlations is, of course, uncertain; however, the experiments clearly show that h in Eq. 1 cannot be taken equal to its steady-state value with transients of interest to the engineering applications considered.

The purpose of this report is to present "improved" engineering relationships to account for turbulent forced-convection heat transfer in ducts during flux transients and steady flow. "Engineering relationships" mean mathematical expressions that are sufficiently simple to allow application to engineering analyses in which forced-convection heat transfer may be only one of many interacting physical mechanisms. "Improved" means a demonstrable superiority in accuracy and generality over methods currently in use.

For purposes of illustration, consider the following equation, which applies to a circular tube:

$$t_s - \bar{t} = \frac{q}{h} + \frac{D^2 R_1}{2k\beta}\left(\frac{\partial q}{\partial \ell} + \frac{1}{\bar{u}}\frac{\partial q}{\partial \tau}\right) + \ldots \qquad (3)$$

If the terms represented by "+ ..." are omitted, Eq. 3 will usually account more accurately for turbulent forced-convection heat transfer during heat-flux transients than will Eq. 1. When the heat flux q is independent of time τ, Eq. 3 also serves as an "improved" engineering relation for cases with q a function of axial distance ℓ. Finally, when the heat flux is independent of both time and axial distance, Eq. 3 becomes equivalent to Eq. 1.

Equation 3 is a specialized version of a more general expression. The source of this expression, the approximations upon which it is based, and its potential utility for practical engineering computations, are the major topics of this report.

MAJOR ASSUMPTIONS

The derivation of the general expression of which Eq. 3 is a specialized form is based on a joining of three mathematical techniques previously applied in the literature on forced-convection heat transfer in ducts. The first is the familiar and classic "separation-of-variables" technique for nontransient cases, which leads to related Sturm-Liouville problems and analytical solutions for specific cases in terms of infinite series involving eigenfunctions. The second technique involves (1) integrating such infinite series for cases in which the wall heat flux is treated as an arbitrary continuous function of duct axial distance; and (2) then exploiting the resultant formulation to obtain relatively simple engineering expressions for the general case of axially nonuniform time-independent wall heat flux.[14] The third technique, first used by Siegel[5] for transient plug-flow forced convection in ducts, consists of rephrasing the transient energy equation and its appropriate boundary conditions in terms of a moving-coordinate system (Lagrangian formulation). The combination of the second and third techniques, together with observations concerning the equivalent of thermal entrance regions, leads nearly directly to expressions like Eq. 3.

The usefulness of these three techniques requires certain assumptions, including negligible physical-property temperature dependence, fully developed incompressible flow, and negligible axial heat diffusion. The duct is considered symmetrical in the sense that only one space coordinate normal to the duct axis is required for the mathematical formulation. Turbulent eddy diffusivities for heat transfer must be a function of this coordinate only. All these assumptions have served, for example, as the theoretical basis for justifying the use of Eq. 1 for steady-state heat transfer.

The usefulness of the third technique requires the assumption that the fluid velocity distribution is uniform. This assumption is made here as a reasonable approximation to a highly turbulent flow in a duct. However, the proposed "improved" engineering relationships, like Eq. 3, are formulated so that the assumption does not apply when the heat transfer is steady. The utility of Eq. 2 for transients also requires the same assumption.[2] As a result, the proposed engineering relationships for heat-flux transients, which, in effect, replace Eq. 1 as alternates to use with Eq. 2, are at least consistent with this assumption.

Previous applications of the moving-coordinate system by Siegel,[5] and later by Siegel and Perlmutter,[9] were based on the assumption of negligible turbulent eddy diffusivities in addition to a uniform velocity distribution; i.e., they were based on a plug-flow model for the fluid. However, this additional assumption is not necessary when the flow is steady and, accordingly, is not used here.

The customary use of eddy diffusivities to represent the local diffusion of heat due to turbulence involves another assumption. Eddy diffusivities added to molecular diffusivities imply a time-averaging of turbulent fluctuations. For reasonable application to transients, the time scale of the averaging should be small compared to that of the transients. It is uncertain whether this requirement will be met when fluid temperatures change significantly in time intervals of 1 msec or less. An evaluation of this uncertainty, which also applies to use of Eq. 1, will probably require analyses of carefully designed experiments.

MATHEMATICAL MODEL

In most engineering applications of Eq. 1 involving transients, q, t_s, and $\bar{t}$ are unknown quantities. Further, when q is a function of time, it would be an unusual situation for q not to be a function of position also. In principle, if q were known as a function of time and position, and if initial conditions could be specified, then t_s and $\bar{t}$ would be related to q by integration of the energy equation for the fluid. Thus, a potentially useful mathematical model to study is one in which q is an arbitrary function of both time and position, and initial conditions are sufficiently general to cover more than a few applications of engineering interest.

Detailed considerations are limited to ducts that are symmetrical in the sense previously described. The most general duct shape of this kind is the annular space with different heat fluxes from both walls, represented mathematically so that the circular tube, the infinitely wide parallel plane duct, and an accurate approximation to the rod or pin bundle are included as special cases. This annular space is illustrated in Fig. 1, which also identifies some of the nomenclature to be used. Note that y, the coordinate normal to the duct axis, is chosen as distance from the inner wall of the annular space. This choice is more or less arbitrary, but is convenient for including the limiting cases of circular tubes and parallel-plane ducts in the formulation.

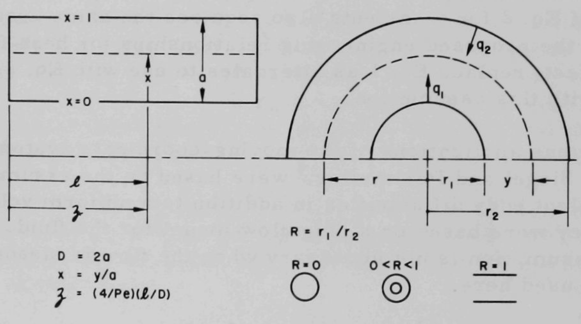

Fig. 1. The Generalized Annular Space

With the major assumptions described previously, the energy equation for the fluid can be written as

$$\frac{1}{r_1 + y}\frac{\partial}{\partial y}\left[(r_1 + y)(\alpha + \epsilon)\frac{\partial t}{\partial y}\right] = u\frac{\partial t}{\partial \ell} + \frac{\partial t}{\partial \tau}, \tag{4}$$

with the boundary conditions

$$-k\left.\frac{\partial t}{\partial y}\right|_{y=0} = q_1(\ell, \tau) \tag{5a}$$

and

$$k \left. \frac{\partial t}{\partial y} \right|_{y=a} = q_2(\ell,\tau). \tag{5b}$$

The temperature at the duct inlet, where $\ell = 0$, is taken as uniform and constant. In Eq. 4, the eddy diffusivity ε and fluid velocity u are known functions of y. As mentioned previously, however, it will be necessary later to use the approximation $u \simeq \bar{u}$.

The initial condition $t(y,\ell,0)$ for Eq. 4 is taken as the steady-state solution corresponding to the time-independent heat fluxes $q_1 = q_1^*(\ell)$ and $q_2 = q_2^*(\ell)$. The formal mathematical solution for this case serves as the main basis for the derivation of the proposed "improved" engineering relations to replace Eq. 1 and will be discussed in some detail. Before doing so, however, Eq. 4 and its associated boundary, inlet, and initial conditions will be recast into a more convenient dimensionless form.

Dimensionless distance normal to the duct axis x ($0 \le x \le 1$), dimensionless axial distance z, and dimensionless time θ are introduced. These and other quantities are defined in the Nomenclature. The dimensionless fluid temperature is denoted by $\xi(x,z,\theta)$, corresponding to dimensionless wall heat fluxes $F_1(z,\theta)$ and $F_2(z,\theta)$. The initial condition is denoted by $\xi(x,z,0) = \xi^*(x,z)$, corresponding to time-independent heat fluxes $F_1^*(z)$ and $F_2^*(z)$. The dimensionless fluid velocity is denoted by $g(x) = u/\bar{u}$.

In addition to the above dimensionless quantities, the second-order differential operator L is introduced for conciseness in notation. This operator is defined by[14]

$$L(\;) \equiv \frac{1}{R + (1-R)x} \frac{\partial}{\partial x} \left\{ [R + (1-R)x] \left(1 + \frac{\varepsilon}{\nu} \mathrm{Pr}\right) \frac{\partial(\;)}{\partial x} \right\}, \tag{6}$$

where R is the radius ratio of the annular space. The circular tube corresponds to $R = 0$; for the parallel-plane duct, $R = 1$.

Finally, the linearity of Eq. 4 and its associated conditions can be taken advantage of by defining

$$\xi(x,z,\theta) = \xi^*(x,z) + \psi(x,z,\theta), \tag{7}$$

where ψ is the dimensionless fluid temperature for the case $F_1^* = F_2^* = 0$, i.e., for the case of initially uniform and constant fluid temperatures.

Use of the above dimensionless quantities and definitions results in the following two related mathematical problems:

For $\xi^*(x,z)$; $0 \le x \le 1$, $z \ge 0$:

$$L\xi^* = g(x)\xi^*_z, \tag{8}$$

$$\xi^*(x,0) = 0, \tag{8a}$$

$$\xi^*_z(0,z) = -F^*_1(z), \tag{8b}$$

and

$$\xi^*_z(1,z) = F^*_2(z). \tag{8c}$$

For $\psi(x,z,\theta)$; $0 \le x \le 1$, $z \ge 0$, $\theta \ge 0$:

$$L\psi = g(x)\psi_z + \psi_\theta, \tag{9}$$

$$\psi(x,z,0) = 0, \tag{10a}$$

$$\psi(x,0,\theta) = 0, \tag{10b}$$

$$\psi_x(0,z,\theta) = -G_1(z,\theta), \tag{10c}$$

and

$$\psi_x(1,z,\theta) = G_2(z,\theta), \tag{10d}$$

where

$$G_i = F_i - F^*_i, \quad i = 1, 2. \tag{11}$$

(Where appearing in the above, and later equations, the subscripts x, z, and θ denote partial differentiation with respect to these variables.)

With the heat-flux functions F_i and F^*_i specified, the foregoing mathematical formulation determines the dimensionless fluid temperature $\xi(x,z,\theta)$. From $\xi(x,z,\theta)$, the dimensionless surface temperatures

$$\xi_{s,1}(z,\theta) \equiv \xi(0,z,\theta), \tag{12}$$

$$\xi_{s,2}(z,\theta) \equiv \xi(1,z,\theta), \tag{13}$$

and bulk temperature

$$\bar{\xi}(z,\theta) = \frac{2}{1+R} \int_0^1 [R + (1-R)\,x]\; g(x)\xi(x,z,\theta)\,dx \tag{14}$$

can also be determined. The intent here, however, is not to specify F_i and F^*_i. Instead, these functions are retained as arbitrary, with the requirement that they be continuous with continuous derivatives. It will be shown that the surface temperatures, bulk temperature, and heat fluxes can then be related to each other in a general way that leads to potentially useful engineering relations of the type given by Eq. 3.

STEADY-STATE NONUNIFORM HEAT FLUX

The solution to this case corresponds to the initial condition $\xi^*(x,z)$, as determined by solution of Eqs. 8. As will be seen, however, it also serves as the basis for determining $\psi(x,z,\theta)$, and hence $\xi(x,z,\theta)$. Accordingly, the superscript "*" will be temporarily dropped from Eqs. 8, and the solution to the general steady-state case will be denoted by $\xi(x,z)$, corresponding to time-independent heat fluxes $F_1(z)$ and $F_2(z)$.

Formal mathematical solutions to this general case involving infinite series of eigenfunctions have been published in various forms; for example, in Refs. 14 and 15. The solution can be written as

$$\xi(x,z) = \bar{\xi}(z) + \sum_{n=1}^{\infty} E_n(x)\lambda_n \int_0^z [C_{1,n}F_1(s) + C_{2,n}F_2(s)]\, e^{-\lambda_n(z-s)}\, ds, \tag{15}$$

where

$$\bar{\xi}(z) = \frac{2}{1+R}\int_0^z [RF_1(s) + F_2(s)]\, ds, \tag{15a}$$

$$C_{1,n} = \frac{2R}{1+R}\,\frac{E_n(0)}{N_n\lambda_n}, \tag{15b}$$

$$C_{2,n} = \frac{2}{1+R}\,\frac{E_n(1)}{N_n\lambda_n}, \tag{15c}$$

and the $E_n(x)$, λ_n, and N_n are eigenfunctions, eigenvalues, and corresponding normalization factors associated with the appropriate Sturm-Liouville problem outlined in the appendix.

In Ref. 14, the equivalent of Eq. 15 was converted into a potentially more useful form by successive integration by parts and rearrangement. These manipulations of Eq. 15 resulted in the heat-flux-independent functions $\zeta_{i,k}(x,z)$, $i = 1, 2$; $k = 1, 2, 3, \ldots$, defined by†

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†The symbol $\zeta_{i,k}$ represents the same function given by $R_{i,k}$ in Refs. 14 and 16. In this report, the symbol $R_{i,k}$ is used for an alternate definition of this function, which has been found to be more convenient for applications.

$$\zeta_{i,k}(x,z) = (-1)^k \sum_{n=1}^{\infty} \frac{C_{i,n}E_n(x)}{\lambda_n^k} e^{-\lambda_n z}. \tag{16}$$

With these functions, Eq. 15 can be written as

$$\xi(x,z) - \bar{\xi}(z) = \sum_{k=0}^{\infty} \left[\zeta_{1,k}(x,0) \frac{d^k F_1(z)}{dz^k} + \zeta_{2,k}(x,0) \frac{d^k F_2(z)}{dz^k} \right]$$

$$- \left[\zeta_{1,k}(x,z) \frac{d^k F_1(0^+)}{dz^k} + \zeta_{2,k}(x,z) \frac{d^k F_2(0^+)}{dz^k} \right]. \tag{17}$$

It can be seen from Eq. 16 that as $z \to \infty$, $\zeta_{i,k}(x,z) \to 0$. As a result, the last bracketed terms of Eq. 17 become negligible for sufficiently large values of z, i.e., for axial distances sufficiently far from the duct inlet. The remaining terms then represent a generalized, fully developed heat-transfer condition in the sense that any remaining z dependence occurs only because of the flux distribution and is easily accounted for once the functions $\zeta_{i,k}(x,0)$, which are independent of z and the flux distributions, are known.

In most engineering applications, only the surface temperatures $\xi(0,z)$ and $\xi(1,z)$ are required, and only the fully developed heat-transfer condition in the sense described above need be considered. As a result, values of the functions $\zeta_{i,k}(x,0)$ at $x = 0$ and $x = 1$ were given special significance in Refs. 14 and 16.† In fact, it is easily shown that

$$\zeta_{1,0}(0,0) = 2(Nu_{1,1})^{-1}, \tag{18a}$$

$$\zeta_{2,0}(0,0) = 2[(Nu_{1,2})^{-1} - (Nu_{1,1})^{-1}], \tag{18b}$$

$$\zeta_{2,0}(1,0) = 2(Nu_{2,1})^{-1}, \tag{18c}$$

and

$$\zeta_{1,0}(1,0) = 2[(Nu_{2,2})^{-1} - (Nu_{2,1})^{-1}], \tag{18d}$$

where the $Nu_{i,j}$ represent fully developed uniform-flux Nusselt numbers, with $i = 1$ referring to the inner wall of the annular space, and $i = 2$ referring to the outer wall. The subscript $j = 1$ denotes the case of a nonzero heat flux from the "i" wall only, and $j = 2$ denotes the case of equal heat

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†See footnote on previous page.

fluxes from both walls. For the circular tube, $F_1 = 0$ and only the $\zeta_{2,k}(1,0)$ are needed. For the parallel-plane duct, symmetry requires that $\zeta_{1,k}(0,0) = \zeta_{2,k}(1,0)$ and $\zeta_{1,k}(1,0) = \zeta_{2,k}(0,0)$.

The relationships given by Eqs. 18 are particularly important, since the Nusselt numbers $Nu_{i,j}$ are those easiest to determine by experiment and include those usually correlated by available empirical equations for forced convection in ducts.

With ξ replaced by ξ^*, and F_i replaced by F_i^*, Eq. 17 can be used to determine ξ^* in Eq. 6. It will now be shown that Eq. 17 can also be used to determine ψ in Eq. 6.

TRANSIENT HEAT FLUXES

The solution to this general case requires the dimensionless transient temperature functions $\psi(x,z,\theta)$, as defined by Eq. 7. This function is determined by solution of the mathematical problem specified by Eqs. 9 and 10. At this point in the analysis the assumption $u \simeq \bar{u}$ (i.e., $g(x) = 1$) is made as a reasonable approximation to highly turbulent flows. For then, each element of fluid moves at the same speed, and by a change to a moving coordinate system, Eqs. 9 and 10 can be made to correspond to Eqs. 8. In effect, either z or θ can be eliminated as an independent variable in the differential equation and its boundary conditions. As a result, Eq. 17, when properly interpreted, can be used to represent $\psi(x,z,\theta)$.

At present, the errors introduced by this uniform-fluid-velocity approximation are uncertain. However, they are expected to be small, especially with liquid metals, for which thermal resistances are not localized near the duct walls as they are with high-Prandtl-number fluids. Also, as mentioned previously, this approximation is required to justify the transient one-dimensional-energy equation (e.g., Eq. 2);[2] for example, by simple integration of Eq. 4 with respect to y. Since the major objective here is to propose potentially more accurate alternates to Eq. 1 for use with the one-dimensional-energy equation, the approximation is at least consistent with this usage.

The change to a moving-coordinate system follows the publications of Siegel[5] and Siegel and Perlmutter.[9] The reader is referred to these publications for further descriptive details.

For $\theta \le z$, the independent variable z is replaced by $z = z_0 + \theta$. The new variable, z_0, represents the dimensionless axial location of a "slice" of fluid at the time the transient begins, i.e., at $\theta = 0$. For a particular value of z_0, $z = z_0 + \theta$ represents a moving coordinate following a particular "slice" of fluid as it moves through the duct. The change in independent variables from (x,z,θ) to (x,θ,z_0), when applied to Eqs. 9 and 10 with $g(x) = 1$, results in

$$L\psi^- = \psi_\theta^-, \tag{19}$$

$$\psi^-(x,0) = 0, \tag{20a}$$

$$\psi_x^-(0,\theta) = -G_1^-(\theta), \tag{20b}$$

and

$$\psi^-(1,\theta) = G_2^-(\theta), \tag{20c}$$

where z_0 is an arbitrary parameter.

The superscript minus sign is used to emphasize that the above equations apply only when $\theta \le z$. Further, since z_0 serves only as an arbitrary parameter held constant for the derivatives of Eq. 19, there are only two truly independent variables: x and θ. In effect, z has been eliminated as an independent variable; and to emphasize this, $\psi(x,z,\theta) = \psi(x, z_0+\theta, \theta)$ is represented by $\psi^-(x,\theta)$, and $G_i(z,\theta) = G_i(z_0+\theta, \theta)$ is represented by $G_i^-(\theta)$. As a result, the correspondence of Eqs. 19 and 20, to Eqs. 8 with $g(x) = 1$ is easily recognized when ξ^* is replaced by ψ^-, F_i^* is replaced by G_i^-, and z is replaced by θ. Thus, the solution for $\psi^-(x,\theta)$ can be obtained from Eq. 17 as

$$\psi^-(x,\theta) - \bar{\psi}^-(\theta) = \sum_{k=0}^{\infty}\left[\zeta'_{1,k}(x,0)\frac{d^k}{d\theta^k}G_1^-(\theta) + \zeta'_{2,k}(x,0)\frac{d^k}{d\theta^k}G_2^-(\theta)\right]$$
$$- \left[\zeta'_{1,k}(x,\theta)\frac{d^k}{d\theta^k}G_1^-(0) + \zeta'_{2,k}(x,\theta)\frac{d^k}{d\theta^k}G_2^-(0)\right], \tag{21}$$

with $z = z_0 + \theta$, $\theta \le z$, and

$$\bar{\psi}^-(\theta) = \frac{2}{1+R}\int_0^\theta [RG_1^-(s) + G_2^-(s)]\, ds. \tag{22}$$

The primes attached to the functions $\zeta_{i,j}$ signify that they should be computed with $g(x) = 1$ (see Eq. 16 and the appendix) in order for Eq. 21 to be an exact solution for Eqs. 19 and 20.

Equation 21 can be rewritten in terms of the independent variables z and θ by evaluating the derivatives $d^kG_i^-/d\theta^k$, noting that these are total derivatives for constant z_0. It is easily shown that

$$\frac{dG_i^-}{d\theta} = \frac{\partial G_i}{\partial \theta} + \frac{\partial G_i}{\partial z}$$

and

$$\frac{d^2G_i^-}{d\theta^2} = \frac{\partial^2 G_i}{\partial \theta^2} + 2\,\frac{\partial^2 G_i}{\partial \theta \partial z} + \frac{\partial^2 G_i}{\partial z^2},$$

and that in general, using operator notation,

$$\frac{d^kG_i^-}{d\theta^k} = \left(\frac{\partial}{\partial \theta} + \frac{\partial}{\partial z}\right)^k G_i \equiv D^kG_i. \tag{23}$$

Thus, Eq. 21 can be rephrased as

$$\psi(x,z,\theta) - \bar{\psi}(z,\theta) = \sum_{k=0}^{\infty} \left[\zeta_{1,k}'(x,0)D^kG_1(z,\theta) + \zeta_{2,k}'(x,0)D^kG_2(z,\theta)\right]$$

$$- \left[\zeta_{1,k}'(x,\theta)D^kG_1(z_0,0) + \zeta_{2,k}'(x,\theta)D^kG_2(z_0,0)\right], \tag{24}$$

with $z_0 = z - \theta \geq 0$.

For $\theta \geq z$, the independent variable θ is replaced by $\theta = \theta_0 + z$. The new variable, θ_0, represents the dimensionless time when the "slice" of fluid at axial position z was at the duct inlet. The procedure is the same as that described above for $\theta < z$, except that θ_0 now serves as an arbitrary parameter and the change in independent variables is from (x,z,θ) to (x,z,θ_0). Equations 9 and 10 with $g(x) = 1$ become

$$L\psi^+ = \psi_z^+, \tag{25}$$

$$\psi^+(x,0) = 0, \tag{26a}$$

$$\psi_x^+(0,z) = -G_1^+(z), \tag{26b}$$

and

$$\psi_x^+(1,z) = G_2^+(z), \tag{26c}$$

where θ_0 is an arbitrary parameter, with

$$\psi(x,z,\theta) = \psi(x,\, z,\, \theta_0 + z) \equiv \psi^+(x,z),$$

and

$$G_i(z,\theta) = G_i(z,\, \theta_0 + z) \equiv G_i^+(z).$$

The form of solution corresponding to Eqs. 21 and 22 can be obtained by simply replacing θ by z in these equations, noting, of course, that $\theta = \theta_0 + z$ and $\theta \geq z$. The form of solution corresponding to Eq. 24 then follows by evaluating the total derivatives $d^k G_i{}^+/dz^+$ in terms of the independent variables z and θ. Thus,

$$\psi(x,z,\theta) - \overline{\psi}(z,\theta) = \sum_{k=0}^{\infty} \left[\zeta'_{1,k}(x,0) D^k G_1(z,\theta) + \zeta'_{2,k}(x,0) D^k G_2(z,\theta)\right]$$

$$- \left[\zeta'_{1,k}(x,z) D^k G_1(0,\theta_0) + \zeta'_{2,k}(x,z) D^k G_2(0,\theta_0)\right], \qquad (27)$$

with $\theta_0 = \theta - z \geq 0$.

Equations 24 and 27 are more general equivalents of the relationships derived by Siegel and Perlmutter[9] for flux transients with plug flow in parallel-plane ducts. These equations differ in form primarily because of the use of the functions $\zeta'_{i,k}$. They are more general, because turbulent diffusivities are allowed to depend on x, and duct shape is more inclusive. With the functions $\zeta'_{i,k}$ known, which requires knowledge of the required eigenfunctions and eigenvalues from the appropriate Sturm-Liouville problem (see appendix), Eqs. 24 and 27 can be applied to obtain solutions for various cases with $G_i(z,\theta)$ specified. As mentioned previously, however, solutions for specified heat flux cases are not the objective here. Instead, Eqs. 24 and 25 will now be used to infer potentially useful engineering relationships.

ENGINEERING RELATIONSHIPS FOR FLUX TRANSIENTS

The solution for $\xi^*(x,z)$, as given by Eq. 17 with $\xi = \xi^*$ and $F_i = F_i^*$, can now be added to the solution for $\psi(x,z,\theta)$, as given by Eqs. 24 and 27. According to Eq. 7, this addition of solutions results in generalized expressions for $\xi(x,z,\theta)$ for the two regions $\theta \leq z$ and $\theta \geq z$. But Eqs. 24 and 27 require the approximation $u \simeq \overline{u}$ (i.e., $g(x) = 1$), and Eq. 17 does not. As a result, the functions $\zeta'_{i,k}$ and $\zeta_{i,k}$ need not be equivalent. If these functions are not equal, then the expressions for $\xi(x,z,\theta)$ will involve terms like $\zeta'_{i,k} F_i + (\zeta_{i,k} - \zeta'_{i,k}) F_i^*$; clearly, equality of these functions is desirable for simplicity.

The functions $\zeta'_{i,k}$ and $\zeta_{i,k}$ can be made equivalent by using the uniform-fluid-velocity approximation for both ξ^* and ψ. Although errors introduced by this approximation are expected to be small, especially for liquid metals, they can be completely suppressed for conditions in which $\psi(x,z,\theta)$ eventually becomes time-independent. This can be accomplished by not using the uniform fluid velocity for $\zeta_{i,k}$ and then simply taking $\zeta'_{i,k} = \zeta_{i,k}$. This is the approach adopted here.

For sufficiently large values of θ, the last bracketed terms of Eq. 24 become negligibly small. Similarly, for sufficiently large values of z, the equivalent terms in Eq. 27 also become negligibly small. As a result, when both θ and z are sufficiently large, Eqs. 24 and 27 are the same, and the parameters z_0 and θ_0 need not be considered. This is a further generalization of the fully developed heat-transfer condition discussed previously with respect to Eq. 17, but applied now to transient as well as nonuniform heat fluxes. Thus, for $\zeta'_{i,k} = \zeta_{i,k}$ and for both z and θ sufficiently large, the solution for $\xi(x,z,\theta)$ can be written

$$\xi(x,z,\theta) - \bar{\xi}(z,\theta) = \sum_{k=0}^{\infty} \zeta_{1,k}(x,0)D^k F_1(z,\theta) + \zeta_{2,k}(x,0)D^k F_2(z,\theta). \quad (28)$$

The dimensionless axial distance z and dimensionless time θ have been defined in a way convenient for discussing the derivation of Eq. 28. The definitions of the functions $\zeta_{i,k}$ that result, however, are relatively inconvenient for applications; in particular, they depend heavily on the index k. Except for sign, the k dependence can be nearly removed by redefining dimensionless distance and time by

$$Z = \lambda_1 z = \beta(\ell/D) \quad (29a)$$

and

$$\Theta = \lambda_1 \theta = \beta(\bar{u}\tau/D), \quad (29b)$$

with

$$\beta = 4\lambda_1/Pe, \quad (30)$$

where λ_1 is the least nonzero eigenvalue from the appropriate Sturm-Liouville problem (see appendix). When the F_i in Eq. 28 are considered to be functions of Z and Θ, rather than z and θ, the functions $\zeta_{i,k}$ are, in effect, multiplied by λ_1^k.

Equation 28 is now specialized for the surface temperatures $\xi(0,z,\theta)$ and $\xi(1,z,\theta)$, and rewritten in a more convenient dimensional form, using Eqs. 18 and the new variables Z and Θ. For this purpose, $T_{s,i}$ is used to represent the surface temperature of either the inner ($i = 1$) or outer ($i = 2$) wall of the annular space. Also used are the fully developed, uniform-heat-flux heat-transfer coefficients $h_{i,j}$, corresponding to the Nusselt numbers of Eqs. 18 and new important coefficients denoted by $R_{i,k}$ and $S_{i,k}$. With the deviative operator D^k defined with respect to Z and Θ, the specialized form of Eq. 28 can be written as

$$t_{s,i} - \bar{t} = \frac{q_i - q_j}{h_{i,1}} + \frac{q_j}{h_{i,2}} + \frac{D}{2k} \sum_{k=1}^{\infty} R_{i,k}D^k q_i + S_{i,k}D^k q_j, \quad (31)$$

with the understanding that when $i = 1, j = 2$, and when $i = 2, j = 1$. For clarity, $D^k q$ is written out below for $k = 1, 2$, and 3.

For $k = 1$:

$$D^k q = \frac{\partial q}{\partial Z} + \frac{\partial q}{\partial \Theta}$$

$$= \frac{D}{\beta}\left[\frac{\partial q}{\partial \ell} + \frac{1}{\bar{u}}\frac{\partial q}{\partial z}\right]. \tag{32a}$$

For $k = 2$:

$$D^k q = \frac{\partial^2 q}{\partial Z^2} + 2\frac{\partial^2 q}{\partial Z \partial \Theta} + \frac{\partial^2 q}{\partial^2 \Theta}$$

$$= \left(\frac{D}{\beta}\right)^2\left[\frac{\partial^2 q}{\partial \ell^2} + \frac{2}{\bar{u}}\frac{\partial^2 q}{\partial \ell \partial \tau} + \frac{1}{\bar{u}^2}\frac{\partial^2 q}{\partial \tau^2}\right]. \tag{32b}$$

For $k = 3$:

$$D^k q = \frac{\partial^3 q}{\partial Z^3} + 3\frac{\partial^3 q}{\partial Z^2 \partial \Theta} + 3\frac{\partial^3 q}{\partial Z \partial \Theta^2} + \frac{\partial^3 q}{\partial \Theta^3}$$

$$= \left(\frac{D}{\beta}\right)^3\left[\frac{\partial^3 q}{\partial \ell^3} + \frac{3}{\bar{u}}\frac{\partial^3 q}{\partial \ell^2 \partial \tau} + \frac{3}{\bar{u}}\frac{\partial^3 q}{\partial \ell \partial \tau^2} + \frac{1}{\bar{u}^3}\frac{\partial^3 q}{\partial \tau^3}\right]. \tag{32c}$$

The $R_{i,k}$ and $S_{i,k}$ are dimensionless coefficients corresponding to the special values of the functions $\zeta_{i,k}$ at $x = 0$ and $x = 1$; in particular,

$$R_{1,k} = \lambda_1^k \zeta_{1,k}(0,0), \tag{33a}$$

$$S_{1,k} = \lambda_1^k \zeta_{1,k}(1,0), \tag{33b}$$

$$R_{2,k} = \lambda_1^k \zeta_{2,k}(1,0), \tag{33c}$$

and

$$S_{2,k} = \lambda_1^k \zeta_{2,k}(0,0), \tag{33d}$$

$$= RS_{1,k}. \tag{33e}$$

These coefficients and β are functions of only Reynolds (or Peclét) and Prandtl numbers, and of duct shape. For sufficient large values of k, their absolute values become independent of k. Also, the $R_{i,k}$ are negative for odd values of k and positive for even values of k, while the $S_{i,k}$ are positive for odd values of k and negative for even values of k.

Equation 31, with the infinite summation truncated at k = 1 or 2, is suggested as an "improved" engineering relationship to account for forced-convection heat transfer with heat-flux transients and steady flow. Use of Eq. 31 truncated at $k > 2$ is not recommended, unless the $q_i(\ell,\tau)$ are assumed to be known; at present, the resulting complications appear to be too severe to warrant application to practical engineering computations. Instead, higher-ordered truncated forms of Eq. 31 are suggested as a means of testing accuracy after $q_i(\ell,\tau)$ is determined with k = 1 or 2.

Equation 31, with the summation term suitably truncated, which is suggested as a replacement for Eq. 1, is to be used with a corresponding version of the one (space)-dimensional-energy equation for the fluid. This energy equation can be written as

$$c\rho S\left(\bar{u}\,\frac{\partial \bar{t}}{\partial \ell} + \frac{\partial \bar{t}}{\partial \tau}\right) = A_1 q_1 + A_2 q_2, \tag{34}$$

which is a more general form of Eq. 2, with A_i representing the heat-transfer area per unit duct length for wall "i."

ENGINEERING RELATIONSHIPS FOR PARTICULAR DUCT SHAPES

Applications of Eq. 31 require specialization to the particular duct shape of interest, with corresponding values of the coefficients β, $R_{i,k}$, and $S_{i,k}$. In addition to duct shape, these coefficients are functions of Reynolds and Prandtl numbers. Values of these coefficients over a wide range of Reynolds and Prandtl numbers have been computed for the circular tube, parallel plane duct, and an approximation of a pin or rod bundle. The approximation is the often-used model that considers each pin to be surrounded by an equivalent annular space, with velocities and eddy diffusivities corresponding to a zero-shear condition at a fictitious outer wall. To compute these coefficients, the turbulent velocity g(x) and eddy-diffusivity distribution $\varepsilon(x)$ must be determined first.

Computations of g(x) and $\varepsilon(x)$ were based on the von Kármán-Martinelli universal-velocity-distribution formulas, with eddy diffusivities for heat transfer equal to those for momentum transfer and uniform at their maximum values in the central regions of the duct. The necessary eigenvalues and eigenfunctions were computed by numerical solution of the appropriate Sturm-Liouville problem, using a general computer program developed for such applications.[17] Eigenvalues and eigenfunctions were also computed for comparison with values obtained by others,[18,19] who used different methods for obtaining $g(x)$ and $\varepsilon(x)$, and different numerical procedures. These comparisons were very favorable, indicating, for example, that the use of other methods for computing $g(x)$ and $\varepsilon(x)$ will have only a minor effect on the coefficients β, $R_{i,k}$, and $S_{i,k}$.

Circular Tube

Since a circular tube has only one wall, the subscripts i and j in Eq. 31 are unnecessary. Thus, $t_{s,i} = t_s$; $q_i = q$; $q_j = 0$; $h_{i,1} = h$; $R_{i,k} = R_k$; and Eq. 31 can be written as

$$t_s - \bar{t} = \frac{q}{h} + \frac{D}{2k} \sum_{k=1}^{\infty} R_k D^k q \tag{35a}$$

$$= \frac{q}{h} + \frac{D^2 R_1}{2k\beta}\left(\frac{\partial q}{\partial \ell} + \frac{1}{\bar{u}}\frac{\partial q}{\partial \tau}\right) + \dots . \tag{35b}$$

Note that Eqs. 35 identify the source of Eq. 3, and reduce to Eq. 1 when q is independent of ℓ and τ. Values of the coefficients β and R_k are given in Table I. The fully-developed uniform-heat-flux heat-transfer coefficient h can be obtained from available empirical correlations.[20,21]

TABLE I. Values of Coefficients for Circular Tube

Re	Pr	β	$-R_1$	R_2	$-R_3$
10^4	0.00316	1.942	0.164	0.149	0.145
	0.01	0.727	0.146	0.131	0.128
	0.0316	0.343	0.110	0.978	0.949
	1.0	0.166	0.0104	0.00900	0.00862
3.16×10^4	0.00316	0.706	0.147	0.133	0.130
	0.01	0.319	0.113	0.102	0.0992
	0.0316	0.195	0.0674	0.0602	0.0583
	1.0	0.138	0.00368	0.00327	0.00315
10^5	0.00316	0.302	0.117	0.106	0.103
	0.01	0.178	0.0722	0.0648	0.0628
	0.0316	0.138	0.0334	0.0299	0.0289
	1.0	0.119	0.00135	0.00120	0.00116
3.16×10^5	0.00316	0.165	0.0764	0.0688	0.0668
	0.01	0.124	0.0364	0.0326	0.0316
	0.0316	0.112	0.0138	0.0124	0.0120
	1.0	0.106	0.000482	0.000432	0.000417
10^6	0.00316	0.114	0.0392	0.0352	0.0341
	0.01	0.101	0.0151	0.0136	0.0131
	0.0316	0.0968	0.00516	0.00463	0.00448
	1.0	0.0949	0.000169	0.000152	0.000147

Parallel-plane Duct

The two walls for this case are geometrically equivalent. As a result, $R_{1,k} = R_{2,k} = R_k$; $S_{1,k} = S_{2,k} = S_k$; $h_{1,1} = h_{2,1} = h_1$; $h_{1,2} = h_{2,2} = h_2$; and Eq. 31 can be written as

$$t_{s,i} - \bar{t} = \frac{q_i - q_j}{h_1} + \frac{q_j}{h_2} + \frac{D}{2k} \sum_{k=1}^{\infty} \left(R_k D^k q_i + S_k D^k q_j \right) \tag{36a}$$

$$= \frac{q_i - q_j}{h_1} + \frac{q_j}{h_2} + \frac{D^2 R_1}{2k\beta} \left(\frac{\partial q_i}{\partial \ell} + \frac{1}{\bar{u}} \frac{\partial q_i}{\partial \tau} \right) + \frac{D^2 S_1}{2k\beta} \left(\frac{\partial q_j}{\partial \ell} + \frac{1}{\bar{u}} \frac{\partial q_j}{\partial \tau} \right) + \ldots . \tag{36b}$$

When q_1 and q_2 are independent of ℓ and τ, Eqs. 36 reduce to the more accurate equivalents of Eq. 1 given in Refs. 14 and 16. Values of the coefficients β, R_k, and S_k are listed in Table II. Recall that h_1 and h_2 represent fully-developed uniform-heat-flux heat-transfer coefficients, with h_1 corresponding to the case of $q_i \neq 0$, $q_j = 0$, and h_2 corresponding to the case of $q_i = q_j$. Values of h_1 and h_2 can be obtained from available correlations.[20,21]

TABLE II. Values of Coefficients for Parallel-plane Duct

Re	Pr	β	$-R_1$	R_2	$-R_3$	S_1	$-S_2$	S_3
10^4	0.00316	1.483	0.196	0.183	0.180	0.168	0.176	0.178
	0.01	0.518	0.179	0.167	0.164	0.153	0.160	0.162
	0.0316	0.213	0.141	0.130	0.128	0.119	0.125	0.127
	1	0.0757	0.0163	0.0138	0.0133	0.0119	0.0128	0.0130
3.16×10^4	0.00316	0.492	0.187	0.174	0.171	0.160	0.167	0.170
	0.01	0.196	0.151	0.140	0.138	0.128	0.135	0.136
	0.0316	0.102	0.0950	0.0873	0.0857	0.0794	0.0835	0.0847
	1	0.0601	0.00603	0.00522	0.00508	0.00461	0.00490	0.00499
10^5	0.00316	0.185	0.159	0.147	0.145	0.135	0.142	0.143
	0.01	0.0931	0.103	0.0952	0.0953	0.0868	0.0911	0.0924
	0.0316	0.0639	0.0501	0.0455	0.0446	0.0411	0.0433	0.0440
	1	0.0507	0.00221	0.00194	0.00189	0.00172	0.00182	0.00186
3.16×10^5	0.00316	0.0865	0.111	0.102	0.100	0.0930	0.0976	0.0990
	0.01	0.0576	0.0552	0.0503	0.0492	0.0454	0.0478	0.0486
	0.0316	0.0484	0.0218	0.0196	0.0191	0.0175	0.0185	0.0189
	1	0.0443	0.000804	0.000705	0.000686	0.000623	0.000662	0.000675
10^6	0.00316	0.0527	0.0599	0.0546	0.0535	0.0494	0.0521	0.0529
	0.01	0.0436	0.0242	0.0217	0.0212	0.0194	0.0205	0.0209
	0.0316	0.0407	0.00850	0.00753	0.00734	0.00669	0.00710	0.00723
	1	0.0394	0.000287	0.000251	0.000244	0.000222	0.000236	0.000240

Pin or Rod Bundle

Since circumferential variations of temperature and heat flux are to be expected in a pin bundle, and since such variations were not considered in the derivation of Eq. 31, the surface temperature, wall heat flux, and fully developed heat-transfer coefficients that appear in this equation must be considered as circumferential average values. This, of course, applies also to current usage of Eqs. 1 and 2 for pin or rod bundles.

When heat fluxes from adjacent pins are equal, Eq. 31 becomes equivalent in form to Eqs. 35 for the circular tube. Values of the coefficients β and R_k are given in Table III. These coefficients were computed for pitch-to-diameter ratios of 1.1, 1.3, and 1.5; they were found to be essentially independent of the ratios in this range. However, the model used to approximate the pin or rod bundle[21] is not considered to be very accurate for computing average heat-transfer coefficients when pitch-to-diameter ratios are less than 1.3. As a result, similar inaccuracies are to be expected for the coefficients β and R_k.

TABLE III. Values of Coefficients for Pin Bundles Having Pitch-to-Diameter Ratios of 1.1 to 1.5

Re	Pr	β	$-R_1$	R_2	$-R_3$
10^4	0.00316	2.804	0.181	0.169	0.167
	0.01	0.977	0.163	0.153	0.151
	0.0316	0.400	0.125	0.117	0.115
	1.0	0.137	0.0106	0.00985	0.00967
3.16×10^4	0.00316	0.938	0.172	0.162	0.159
	0.01	0.371	0.136	0.127	0.126
	0.0316	0.192	0.0816	0.0762	0.0751
	1.0	0.110	0.00429	0.00398	0.00391
10^5	0.00316	0.354	0.144	0.135	0.133
	0.01	0.176	0.0897	0.0838	0.0826
	0.0316	0.119	0.0408	0.0381	0.0374
	1.0	0.0929	0.00160	0.00149	0.00146
3.16×10^5	0.00316	0.164	0.0968	0.0905	0.0891
	0.01	0.108	0.0454	0.0423	0.0416
	0.0316	0.0895	0.0169	0.0157	0.0154
	1.0	0.0811	0.000580	0.000538	0.000528
10^6	0.00316	0.0990	0.0496	0.0463	0.0455
	0.01	0.0808	0.0188	0.0174	0.0171
	0.0316	0.0749	0.00633	0.00587	0.00577
	1.0	0.0722	0.000206	0.000191	0.000187

Values of the fully developed, uniform-heat-flux heat-transfer coefficient can be obtained from available empirical correlations.[21]

CONCLUDING REMARKS

The derivation of Eq. 31 is based on a variety of assumptions. Most of these assumptions apply also to Eq. 1 as used in applications; including, for example, the requirement that z be "sufficiently large" and the assumption that $u \simeq \bar{u}$ to justify use of the one (space)-dimensional-energy equation for the fluid, i.e., Eq. 2 or 34. Equation 31 accounts for axially nonuniform and transient wall heat fluxes; Eq. 1 does not. Equation 31 assumes that θ is "sufficiently large," and applications require truncation of the summation term; also, Eq. 31 is based on the approximation $\zeta'(x,z) = \zeta(x,z)$.

The total effect of these assumptions on accuracy of predictions probably cannot be determined in a general quantitative manner. However, a variety of comparative computations for specific cases have been performed. These computations compared predictions, using Eq. 1 and truncated forms of Eq. 31, with the results of calculations based on models that do not use the most critical of these assumptions. For the cases considered, use of Eq. 31, truncated at $k = 1$, usually resulted in improvements in accuracy, compared to use of Eq. 1; and use of Eq. 31, truncated at $k = 2$, always resulted in improvements in accuracy. For conditions of interest to liquid-metal-cooled fast-breeder-reactor safety studies, significant improvements in accuracy appear to result only for extremely rapid transients, e.g., exponential heat-flux increases with exponential periods in the millisecond range. Further comparative computations of this kind are continuing, with emphasis on cases of interest to nuclear-reactor safety studies in general.

There is an interesting correspondence between the computed results obtained by Gopalakrishnan[11] and Eqs. 35 and 36. Recall that Gopalakrishnan treated cases of exponentially increasing heat generation within the duct walls. He found that for sufficiently large time, the "actual" heat-transfer coefficient--i.e., $q/(t_s - \bar{t})$--became independent of time, and that for sufficiently small exponential periods, it was significantly larger than the corresponding fully developed nontransient value. He also found that the "actual" heat-transfer coefficient, when expressed as a Nusselt number, could be treated as a function of Reynolds and Prandtl numbers, and of the dimensionless quantity $D^2/\alpha P$, where P is the exponential period.

With exponentially increasing heat generation, the heat flux to the fluid will eventually become proportional to the heat generation; i.e., $q \sim e^{\tau/P}$. Thus by neglecting the space dependence of q and letting h' denote the "actual" heat-transfer coefficient, Eq. 35b can be rearranged and written

$$\frac{h}{h'} = 1 + \frac{1}{2}\left(\frac{NuR_1}{Pe\beta}\right)\left(\frac{D^2}{\alpha P}\right) + \ldots, \qquad (37)$$

where Nu represents the fully developed uniform-heat-flux Nusselt number. Note that $Pe = RePr$; Nu, R_1, and β are functions only of Re and Pr; and R_1 is negative. Equation 37 predicts all the effects found by Gopalakrishnan, as summarized in the preceding paragraph.

APPENDIX

The Related Sturm-Liouville Problem

The Sturm-Liouville problem which defines the eigenfunctions, eigenvalues, and other quantities for Eq. 15 can be written as

$$LE_n(x) + \lambda_n g(x)E_n(x) = 0, \qquad 0 \le x \le 1, \tag{A.1}$$

with

$$E_n'(0) = E_n'(1) = 0. \tag{A.2}$$

The differential operator L is defined by Eq. 6, and the primes denote differentiation with respect to x. The normalization factor N_n is given by

$$N_n = \frac{2}{1+R} \int_0^1 [R + (1 - R)\,x]\; g(x)E_n^2(x)\; dx. \tag{A.3}$$

The Expansion Coefficients $C_{i,n}$

The $C_{i,n}$ (i = 1 or 2) that appear in Eq. 15 are expansion coefficients for the generalized Fourier expansions of the functions $V_i(x)$. These functions are defined by

$$LV_1(x) = \frac{2R}{1+R}\, g(x), \tag{A.4}$$

with

$$V_1'(0) = -1, \tag{A.5a}$$

and

$$V_1'(1) = 0; \tag{A.5b}$$

and by

$$LV_2(x) = \frac{2}{1+R}\, g(x), \tag{A.6}$$

with

$$V_2'(0) = 0, \tag{A.7a}$$

and

$$V_2'(1) = 1. \tag{A.7b}$$

The $V_i(x)$ are actually dimensionless fluid temperature distributions for fully developed heat transfer in an internally heated ($i = 1$), or externally heated ($i = 2$) annular space, with axially uniform heat flux. The expansion coefficients are given by

$$C_{i,n} = \frac{2}{(1+R)\,N_n} \int_0^1 [R + (1 - R)\,x]\; g(x) E_n(x) V_i(x)\, dx \tag{A.8}$$

$$= \frac{2R}{1 + R} \, \frac{E_n(0)}{N_n \lambda_n} \quad \text{for } i = 1 \tag{A.9a}$$

$$= \frac{2}{1 + R} \, \frac{E_n(1)}{N_n \lambda_n} \quad \text{for } i = 2. \tag{A.9b}$$

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